Parallel Algorithms for $\mathcal{H}$-matrices (Part I)

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4. Matrix-Vector Multiplication

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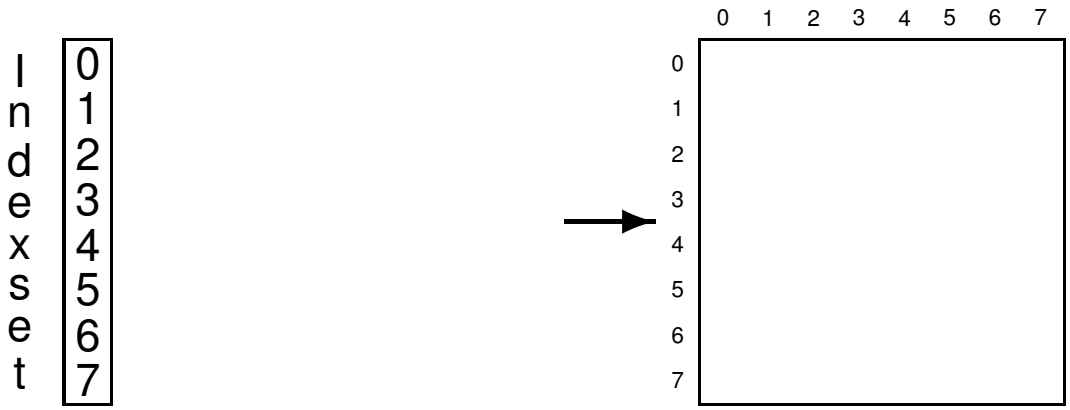

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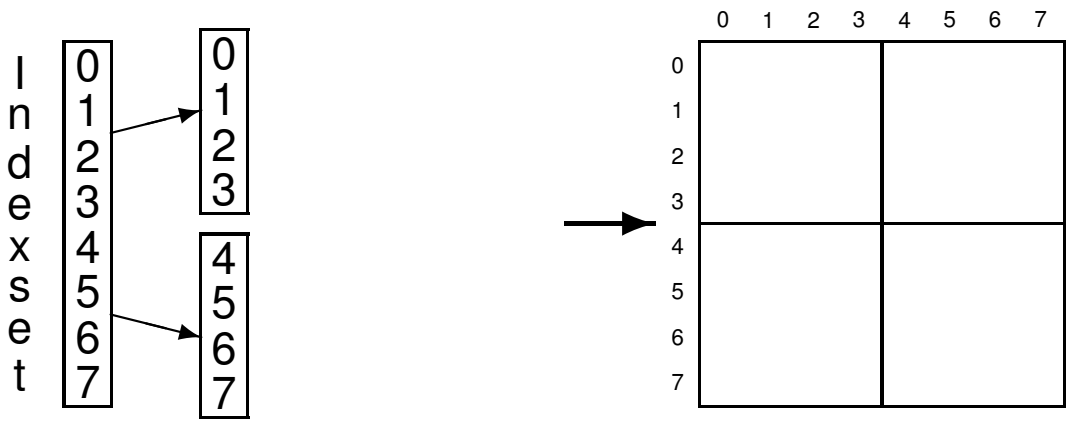

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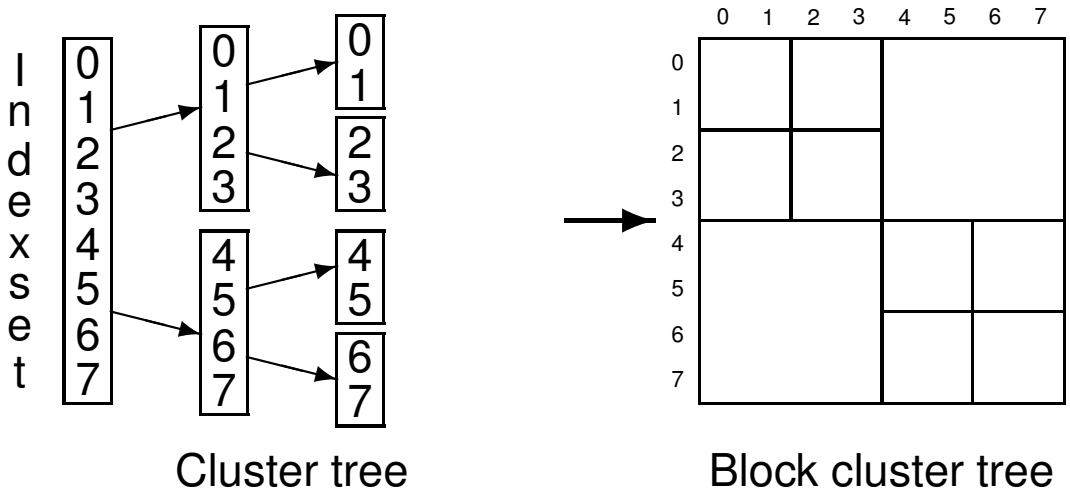

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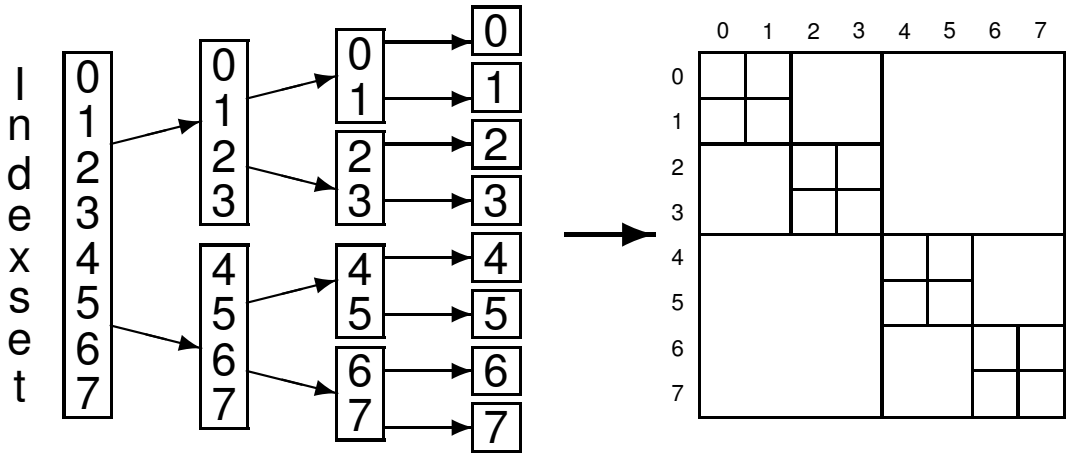

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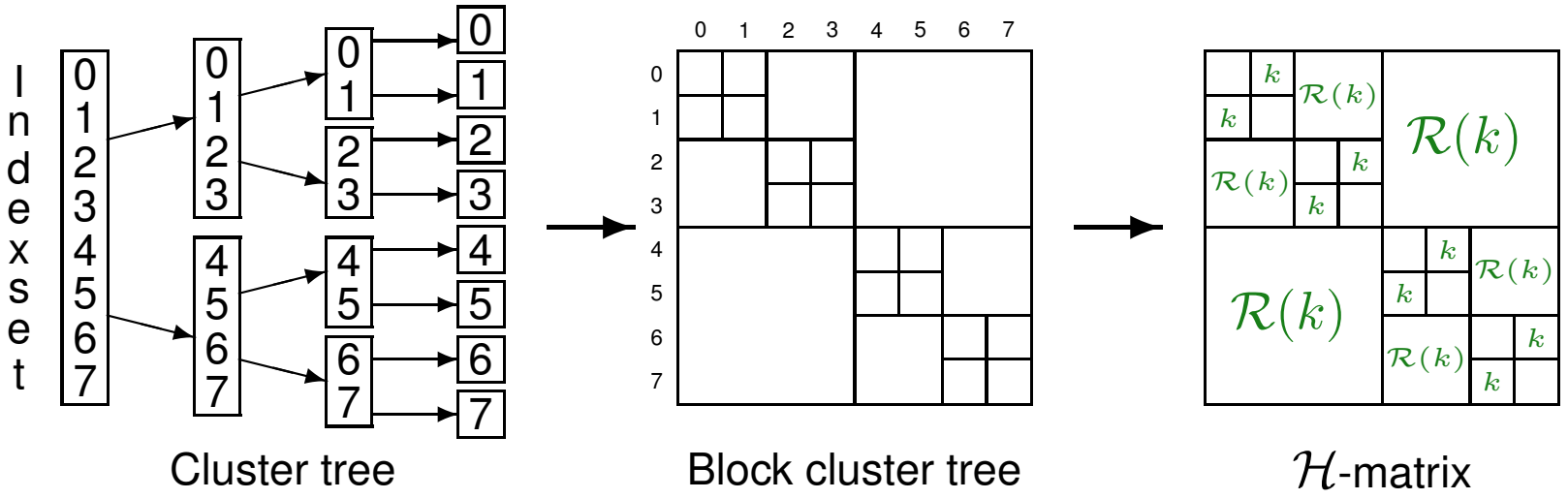

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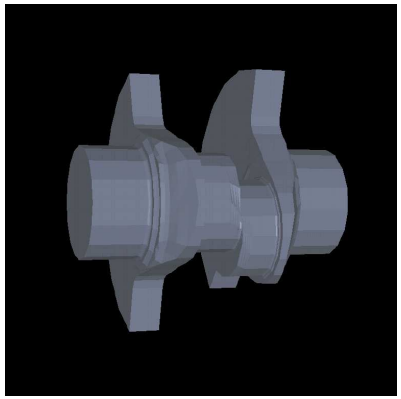

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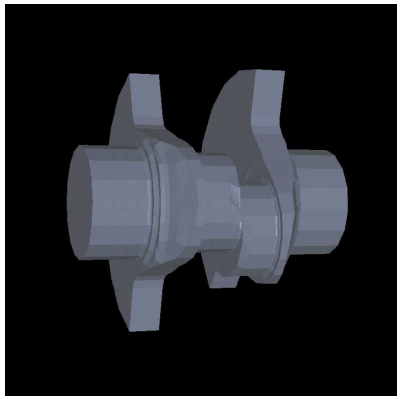

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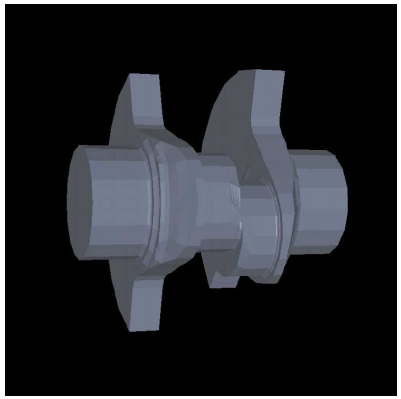

- computed on parallel system with p processors
- Problem chosen because:
  - matrix creation is expensive (compared to FEM)
  - all matrix entries are non-zero (simpler cost function)

Matrix Building

Basic sequential algorithm:

---

for all $(\tau, \sigma) \in \mathcal{L}(T(I \times I))$ **do**
  if (τ, σ) is admissible **then**
    create rank-k matrix;
  else
    create dense matrix;
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Straightforward parallelisation:

create each block on different processor

Matrix Building on Shared Memory Systems

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Parallel Speedup: $\frac{t(1)}{t(p)} \geq \frac{p}{\left(2-\frac{1}{p}\right)}$

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Implementation with *Thread Pool*:

- consists of p threads which execute given jobs
- much simpler interface than Pthreads: simplifies programming
- more efficient: less startup time per job because no real thread is started

Parallel Algorithm with Thread Pool

procedure build_matrix (τ, σ)
    if (τ, σ) is admissible **then**
        build a rank-k matrix using ACA;
    else
        build a dense matrix;
end;

for all $(\tau, \sigma) \in \mathcal{L}(T(I \times I))$ **do**
    run (build_matrix((τ, σ)));
endfor;

sync_all ();

Numerical Results (Shared Memory)

Fixed accuracy ($\varepsilon = 10^{-4}$)

Time and Parallel Efficiency $E(p) = t(1)/(p \cdot t(p))$:

n	$t(1)$	$E(4)$	$E(8)$	$E(12)$	$E(16)$
4 416	54.4 s	99.3 %	98.6 %	98.6 %	94.4 %
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- high efficiency even for small problems
- low overhead of thread pool

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- refined version of list scheduling possible: *Longest Process Time* (*LPT*) scheduling (list scheduling with ordering)
- Parallel Speedup:

$$\frac{t(1)}{t(p)} \geq \frac{p}{\left(\frac{4}{3} - \frac{1}{3p}\right)}$$

LPT algorithm:

---

procedure LPT($\mathcal{L}(T(I \times I))$)

$V = \mathcal{L}(T(I \times I))$;

$C_{0 \leq i < p} = 0$;

while $V \neq \emptyset$ **do**

take $v \in V$, s.t. $c_k(v) = \max_{v \in V} c_k(v)$;

choose $i \in \{0, \ldots, p-1\}$, s.t. $C_i = \min_{0 \leq j < p} C_j$;

assign v to processor i;

$C_i = C_i + c_k(v); V = V \setminus \{v\}$;

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For fixed accuracy approximation:

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- instead choosing an average rank k_{avg} (depending on problem)

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Fixed Rank ($k = 10$)

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Fixed Accuracy ($\varepsilon = 10^{-4}$, $k_{\text{avg}} = 10$)

n	$t(1)$	$E(4)$	$E(8)$	$E(12)$	$E(16)$
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89 412	*2512.0* s	–	*94.0* %	*96.2* %	*91.3* %

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Sequential Systems

Computer model: von Neumann machine

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Hence

- programming for all systems follows same rules
- performance of programs is predictable: by speed of CPU and memory

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No similar model for parallel computers available

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Instead:

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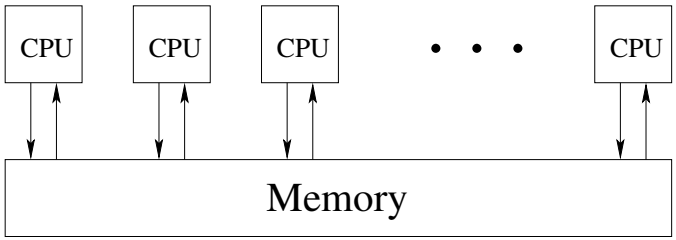
Instead:

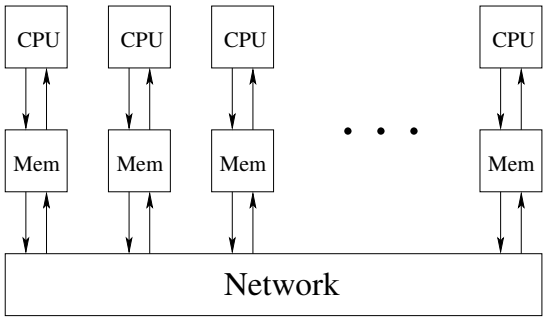
- direct parallelisation: *PVM*, *MPI* etc.
  - Advantage: for many systems available
  - Disadvantage: complicated programming, good knowledge of hardware necessary
- PRAM: p CPUs, global memory shared by all processors



  - Advantage: simple programming, no communication
  - Disadvantage: only for real shared memory systems (not found in practice)

Bulk Synchronous Parallel (BSP) Machine

Most computer systems (shared and distributed memory) have local memory and a network:



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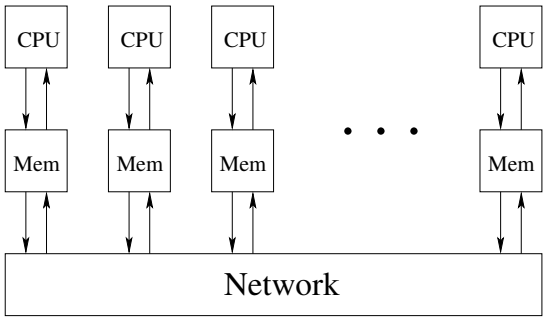


BSP model based on 3 parameters:

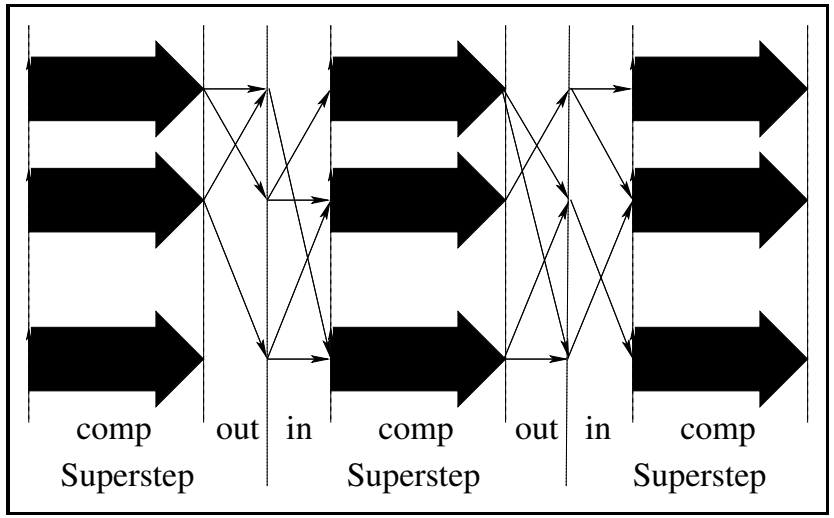
p = the number of processors

g = speed of all CPUs / speed of network ($\sim$ bandwidth)

l = time for a global synchronisation

BSP computation

Sequence of *Supersteps*:



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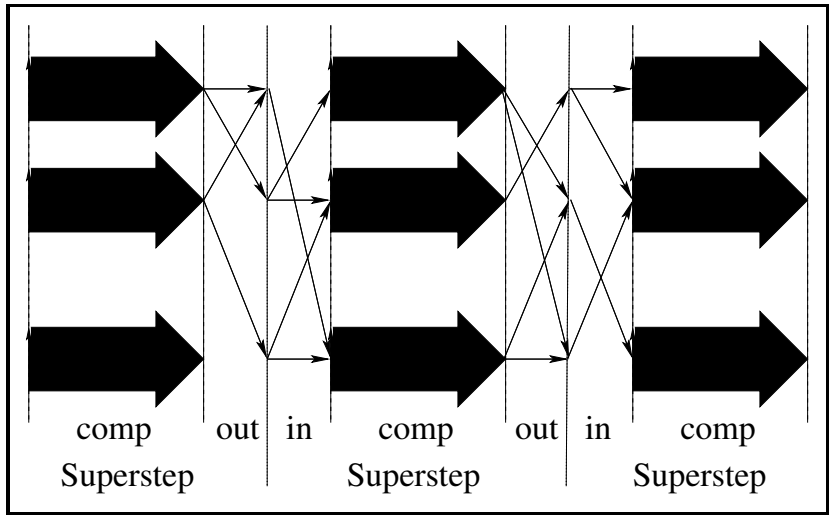

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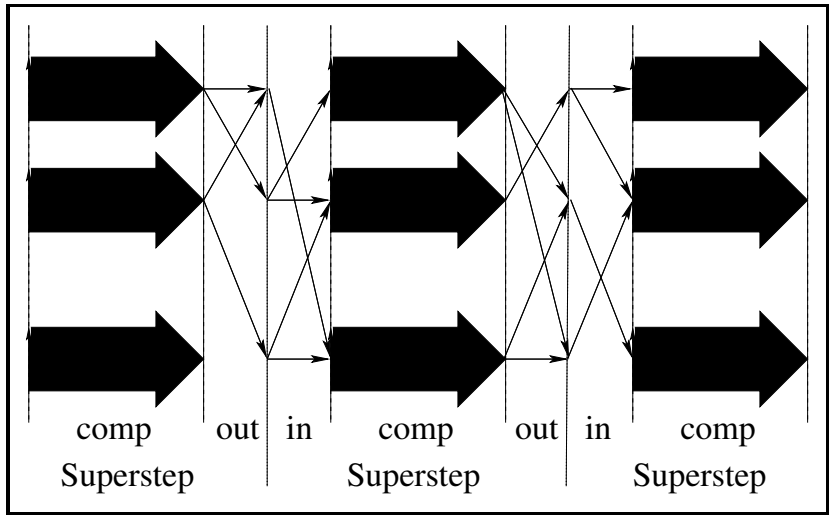

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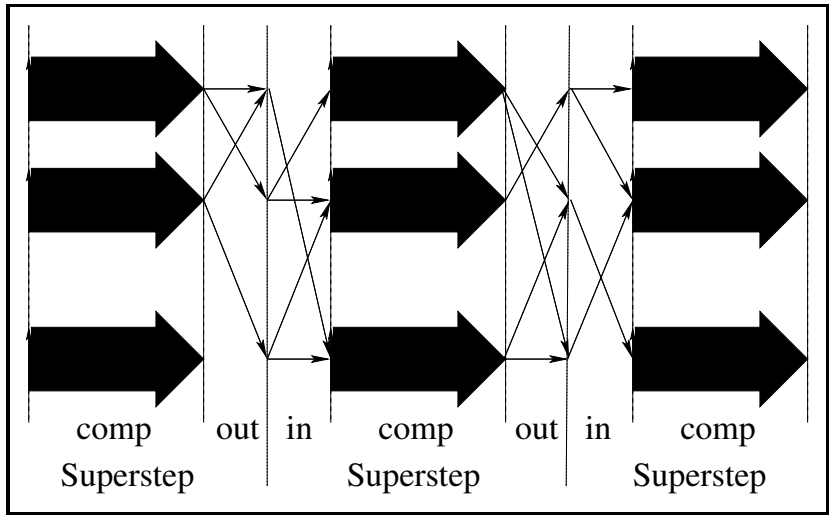

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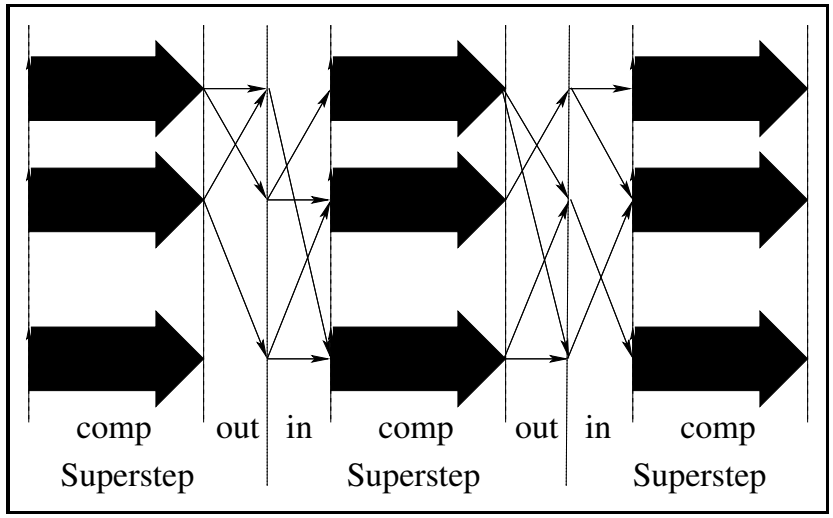

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- for each superstep:
  - amount of work w, maximal size of data sent/received h
  - costs per step: $w + g \cdot h + l$
- summing up S supersteps yields expression:

$$W + g \cdot H + l \cdot S$$

Matrix-Vector Multiplication (Distributed Memory)

Multiplication for Dense Matrices

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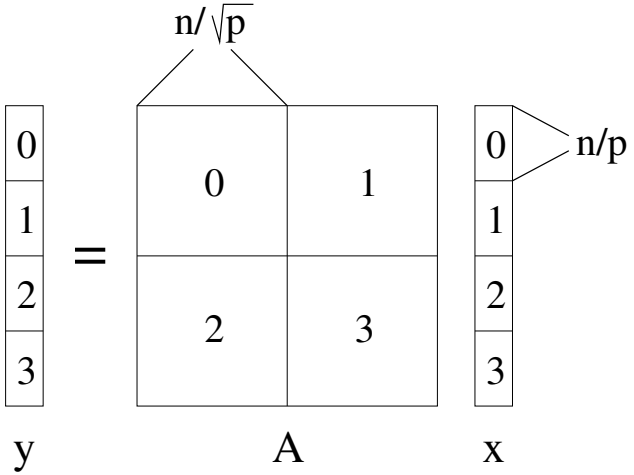
Multiplication for Dense Matrices

considered: $y = \alpha Ax + \beta y$

data distribution (per processor i):

- x_i, y_i: n/p elements
- A: n^2/p elements



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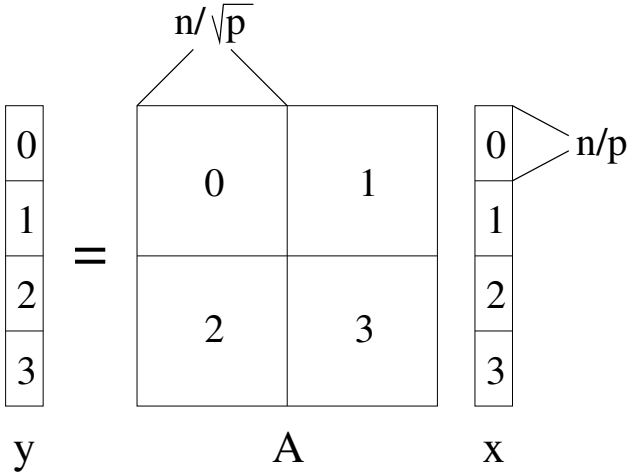

procedure dense_mul(α, A, x, β, y)

{ *first step* }

$y_i = \beta \cdot y_i$;

send x_i to all processors sharing it;

sync();

{ *second step* }

$y'_i = \alpha A x_i$;

send y'_i to all processors sharing it;

sync();

{ *third step* }

$y_i = y_i + \sum y'_j$;

sync();

end;

Complexity of dense Matrix-Vector Multiplication:

$$\mathcal{W}(n,p) = \mathcal{O}\left(\frac{n^2}{p} + \frac{n}{\sqrt{p}}\right) + g \cdot \mathcal{O}\left(\frac{n}{\sqrt{p}}\right) + 3 \cdot l$$

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Multiplication for $\mathcal{H}$-matrices

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$$\mathcal{W}(n,p) = \mathcal{O}\left(\frac{n^2}{p} + \frac{n}{\sqrt{p}}\right) + g \cdot \mathcal{O}\left(\frac{n}{\sqrt{p}}\right) + 3 \cdot l$$

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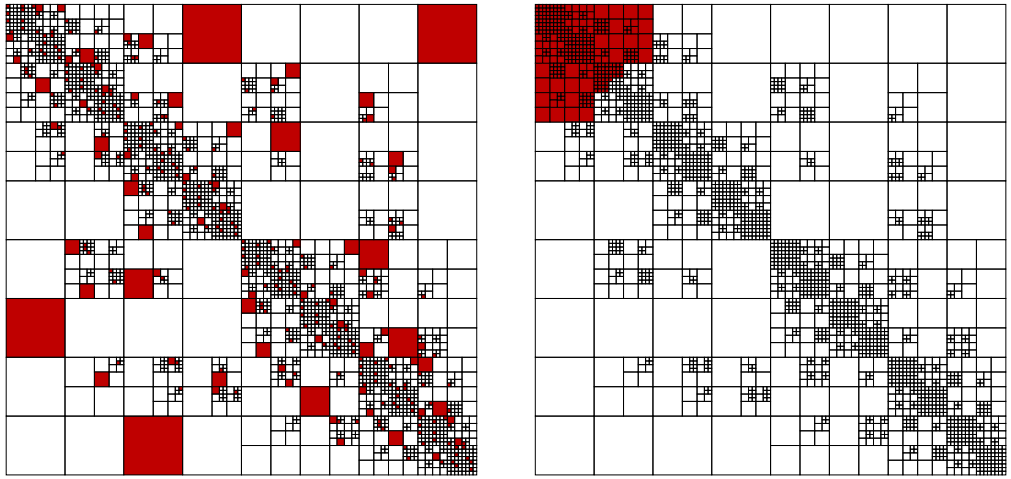

Sharing Constant

Definition: Let $\mathcal{L}(T, q)$ denote the leaves associated to processor q. For $i \in I$ let the *sharing constant* $c_{\text{sh}}(i)$ be defined as

$$c_{\text{sh}}(i) = \max\{c^{\tau}_{\text{sh}}(i), c^{\sigma}_{\text{sh}}(i)\}$$

with

$$\begin{aligned} c^{\tau}_{\text{sh}}(i) &= |\{q \,|\, \exists(\tau, \sigma) \in \mathcal{L}(T, q) : i \in \tau\}| \quad \text{and} \\ c^{\sigma}_{\text{sh}}(i) &= |\{q \,|\, \exists(\tau, \sigma) \in \mathcal{L}(T, q) : i \in \sigma\}|. \end{aligned}$$

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(Rewritten) Complexity of dense Matrix-Vector multiplication ($c_{\text{sh}} = \sqrt{p}$):

$$\mathcal{W}(n, p) = \mathcal{O}\left(\frac{n^2}{p} + \frac{c_{\text{sh}} n}{p}\right) + g \cdot \mathcal{O}\left(\frac{c_{\text{sh}} n}{p}\right) + 3 \cdot l$$

Load Balancing

Goal: minimise c_{sh}.

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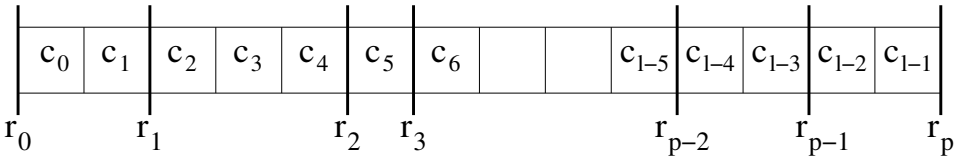
Sequence Partitioning



For (ordered) list of costs $c_0, c_1, \ldots, c_{l-1}$ find set of delimiters $r_0 = 0, \ldots, r_p = l - 1$, s.t.

$$\max_{0 \leq i < p} \sum_{j=r_i}^{r_{i+1}} c_j$$

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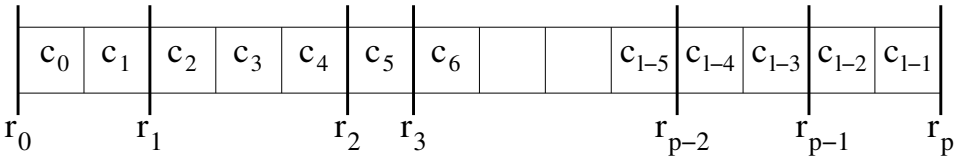

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Sequence partitioning problem can be solved in: $\mathcal{O}(l \cdot p)$.

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Ordering for $\mathcal{L}(T(I \times I))$ is done by *space-filling curves*.

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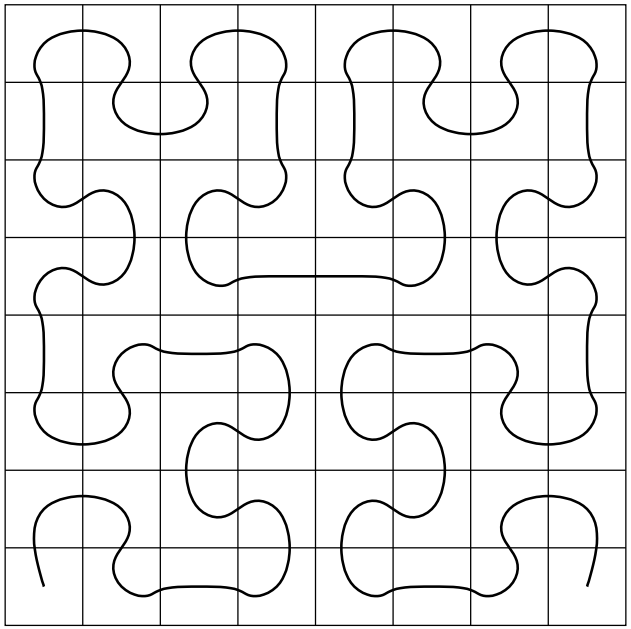

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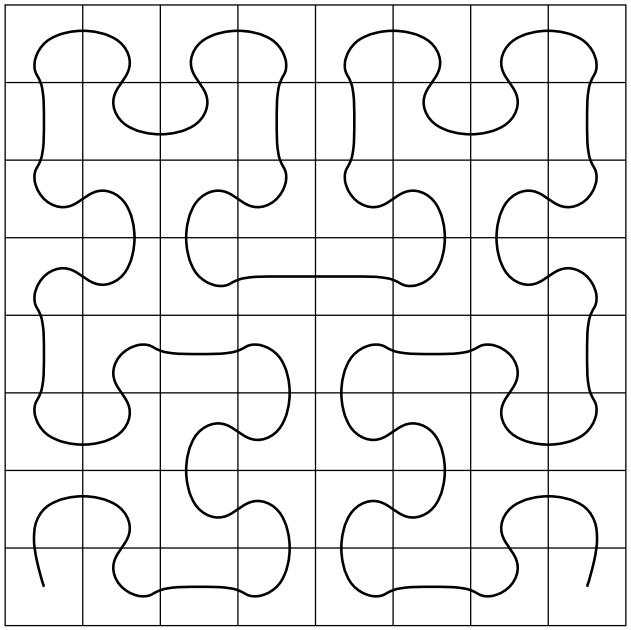
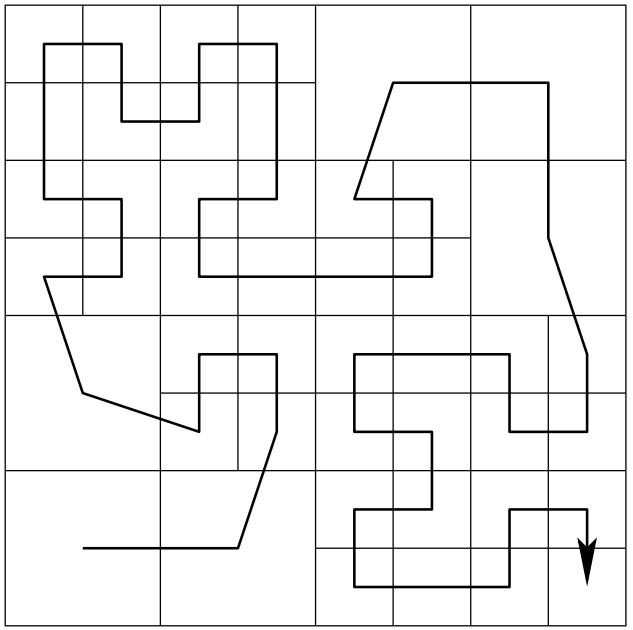


Applied to $\mathcal{L}(T(I \times I))$



Numerical Results

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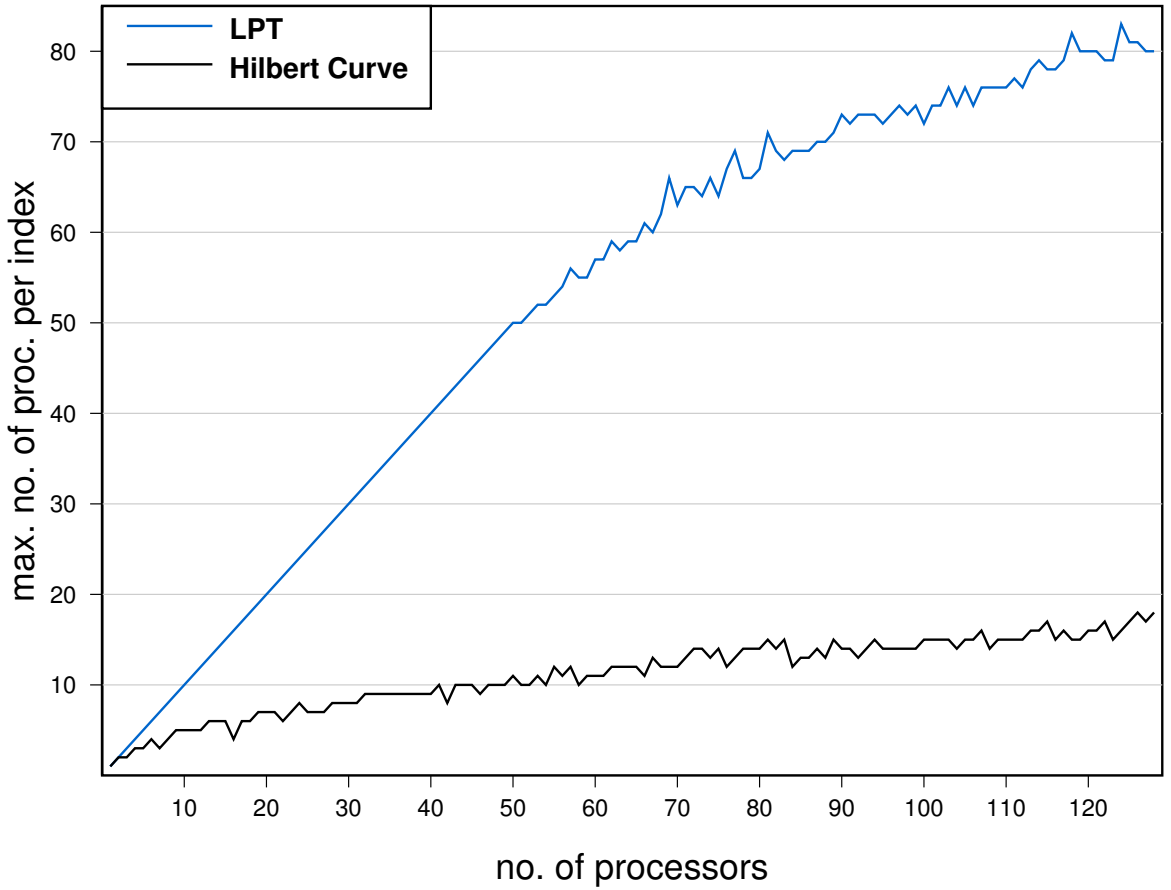
Values of c_{sh}

LPT scheduling vs. sequence part. with space-filling curves:



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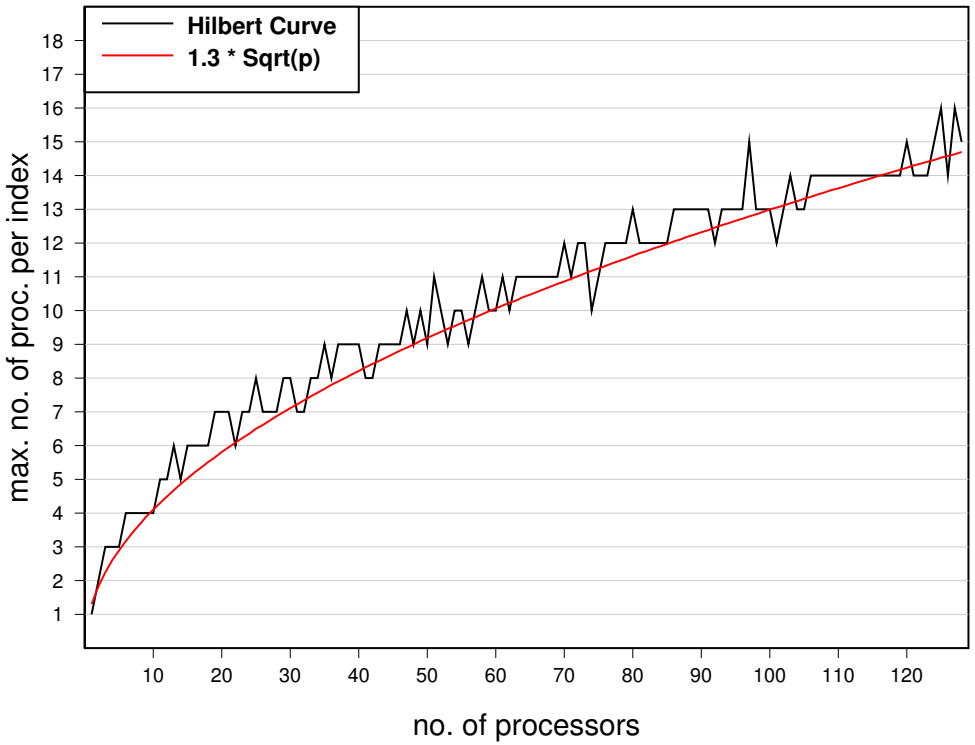

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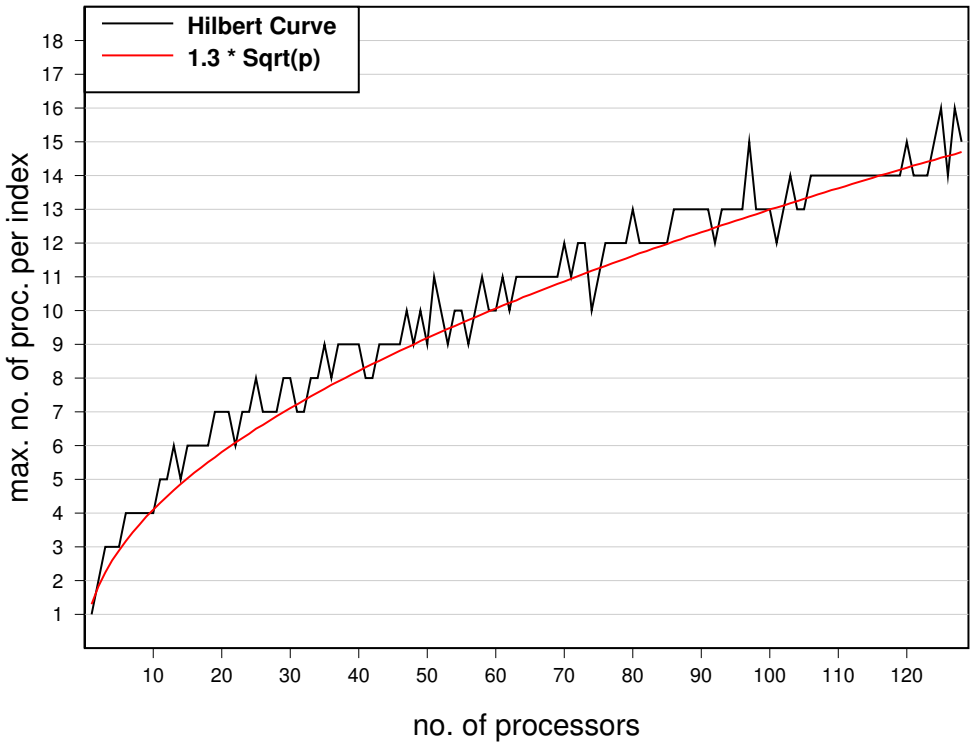

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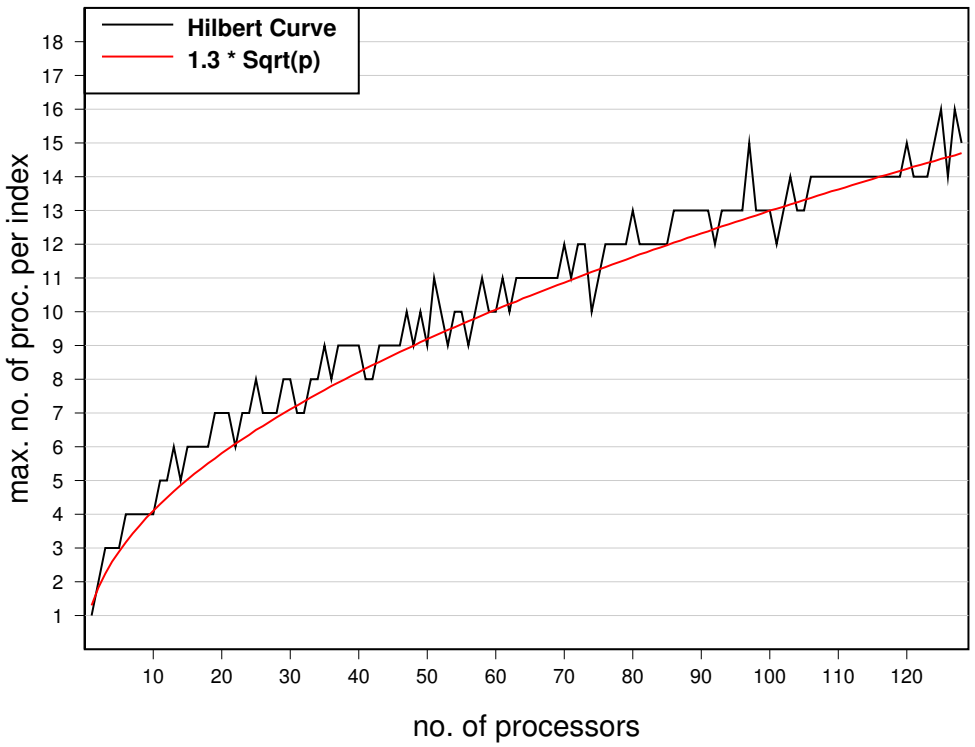

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Time and Parallel Efficiency

Fixed rank ($k = 10$):

n	$t(1)$	$E(4)$	$E(8)$	$E(12)$	$E(16)$
4 416	0.33 s	90.4 %	82.9 %	74.2 %	69.1 %
16 128	1.20 s	87.7 %	84.1 %	79.6 %	76.0 %
89 412	–	–	1.98 s	*97.5* %	*95.9* %

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Fixed accuracy ($\varepsilon = 10^{-4}, k_{\text{avg}} = 10$):

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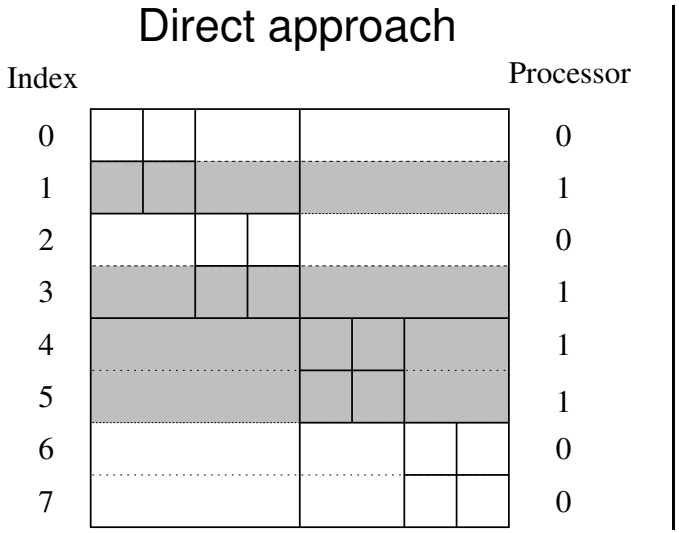

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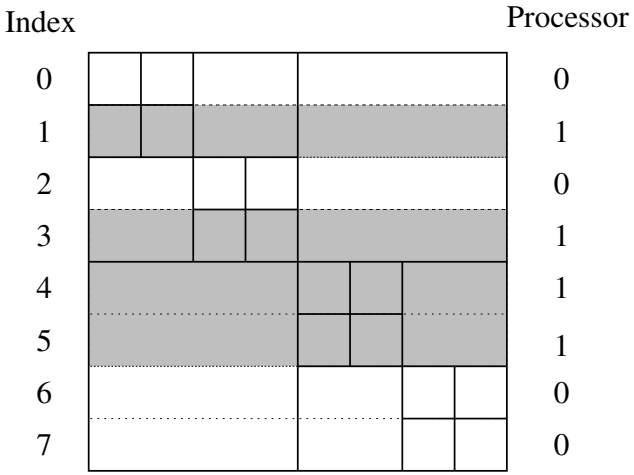
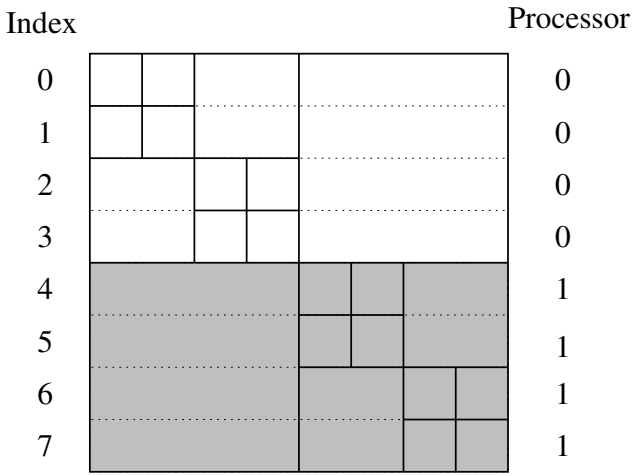
Direct approach



Contiguous index sets



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- use sequence partitioning for cluster tree $T(I)$: contiguous index sets per processor
- use LPT scheduling for partial multiplications

BSP

procedure shm_mul($i, \alpha, A, x, \beta, y$)

{ *step 1* }

$y_i = \beta \cdot y_i$;

send x_i to all processors sharing it;

sync();

{ *step 2* }

compute partial rank-k products on proc i;

send results to all sharing processors;

sync();

{ *step 3* }

$y'_i = \alpha A x_i$;

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{ *step 4* }

$y_i = y_i + \sum y'_j$ (disjoint summation);

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Thread Pool

procedure step_1 (i, β, y, A)

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compute partial products on proc i;

end;

procedure step_2 (i, α, A, x, y)

$y_i = y_i + \alpha A x_i$;

end;

procedure shm_mul_tp($i, \alpha, A, x, \beta, y$)

for $0 \leq i < p$ **do**

run(step_1(i, β, y, A));

sync_all();

for $0 \leq i < p$ **do**

run(step_2(i, α, A, x, y));

sync_all();

end;

Complexity of Shared Memory Multiplication

Complexity of BSP algorithm:

$$\mathcal{W}(n,p) = \mathcal{O}\left(\frac{n \log n}{p}\right) + g \cdot \mathcal{O}(n) + 4 \cdot l$$

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